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LOCAL PRIME DISTANCE ANTIMAGIC LABELING OF GRAPHS

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ABSTRACT
Let $G=(V,E)$ be a graph of order n. Let $f:V(G)\to\{1, 2 \ldots, n\}$ be a perfect mapping. For every node $t\in V(G)$, we define the weight of the node t as $w(t)=\sum_{a\in N(t)} f(a)$ where $N(t)$ is the open neighborhood of the node t. The perfect mapping f is said to be a local prime distance antimagic labeling of G if for every pair of adjacent nodes $s,t\in V(G)$, $w(s)\neq w(t)$ and $\gcd(f(s),f(t))=1$. The local prime distance antimagic labeling f defines a proper node coloring of the graph G, where the node t is assigned the color $w(t)$. We define the local prime distance antimagic chromatic number $\chi_{lp}(G)$ to be the minimum number of colors taken over all colorings induced by local prime distance antimagic labeling of G. In this paper, we introduce the parameter $\chi_{lp}(G)$ and compute the local prime distance antimagic chromatic number of graphs.
Keywords: Local distance antimagic labeling, distance antimagic labeling, local prime distance antimagic labeling, local prime distance antimagic chromatic number.
Mathematics Subject Classification : 05C78

INTRODUCTION:
Initially, graph theory was used to solve amusing mathematical conundrums. This area of interdisciplinary research between Mathematics and other Sciences has grown significantly in the past few decades. A graph is the most powerful type of discrete structure due to its real-world applications and scientific fields. A graph can be represented by a numeric number, a polynomial, a sequence of numbers, or a matrix that represents the entire graph. Graph labeling is a fundamental aspect of graph theory as well as, a mathematical field dedicated to understanding the characteristics and relationships within graphs. It entails the assignment of labels, numeric values assign to the nodes, arcs of a graph according to specific guidelines, restrictions and constraints to construct a graph and its applications in real time scenarios.
This process is highly relevant across diverse disciplines including Computer Science, network analysis, Biology, Chemistry, Social Sciences and so on. The practical applications of graph labeling are extended in network routing, graph visualization, scheduling problems, and code optimization. Moreover, exploration of graph labeling has prompted by the development of inventive algorithms and mathematical approaches which is aimed at addressing challenges within graph theory. The real time applications of graph labeling further extended to social media too.
Let $G=(V,E)$ be a finite, undirected and connected graph with neither loops nor multiple arcs. The ssnode and arc sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. A labeling of a graph G is a mapping that carries a set of graph elements, usually the nodes and arcs into a set of numbers,

usually integers. A node labeling of a graph G is a mapping f from the nodes of G to a set of elements, often integers. Each arc xy has a label that depends on the nodes x and y and their labels $f(x)$ and $f(y)$. If the domain is $V(G) \cup E(G)$ then the labeling is called total labeling.

The motivation and inspiration of this research work are to propose a novel labeling approach to label graphs in a new way, in contrast to other recent strategies, which have mostly been enhancements of established procedures and formulated on specific labelings like prime labeling and local distance antimagic labeling. The authors of this paper keen on formulating a new graph labeling, namely, local prime distance antimagic labeling.

REVIEW OF LITERATURE:

Rosa [15] triggered of a new concept in graph labeling (it was named as β-valuation). The various types of labeling are investigated by Gallian [6]. The concept of the prime labeling originated with Entringer [5] and was introduced by Tout et al. [17]. Hartsfield and Ringel [10] introduced the concept of an antimagic labeling. Distance antimagic labeling was introduced by Arumugam and Kamatchi [1]. Kamatchi and Arumugam [13] proved that the cycle graphs, the wheel graphs and the path graphs are distance antimagic graphs. Kamatchi et al. [14] proved hypercube graphs and several classes of disconnected graphs are distance antimagic. Simanjuntak and Wijaya [16] proved that sun graphs, prism graphs, complete graphs, wheel graphs, fans and friendship graphs are distance antimagic graphs.

Arumugam et al. [2] derived local antimagic labeling and they obtained some basic results. Bensmail et al. [3] proved that every tree other than K_2 is local antimagic. Haslegrave [11] proved that every connected graph other than K_2 is local antimagic. Motivated by this concept, Handa et al. [8] developed the different ideas of a graph labeling which is a local distance antimagic labeling. For more details on this labelings we refer to Handa [7] and Handa et al. [9]. Divya and Yamini [4] introduced the parameter $\chi_{ld}(G)$ and compute the local distance antimagic chromatic number of graphs.

PRELIMINARIES :

Now, graph labeling is defined below:

Definition 3.1 (Rosa [15]) If the nodes or arcs or both of the graph are assigned valued subject to certain conditions, then it is known as graph labeling.

Here, prime labeling is defined below:

Definition 3.2 (Tout et al. [17]) Let $G = (V, E)$ be a graph. If $f : V \to \{1, 2, ..., |V|\}$ is an one-to-one correspondence function. Then f is said to be a prime labeling if for each $e = uv \in E$, we have $\gcd(f(u), f(v)) = 1$. The graph that admits a prime labeling is called a prime graph.

Next, the definition of various types of antimagic labeling techniques is presented to make this paper self-contained.

Definition 3.3 (Hartsfield and Ringel [10]) Let $G = (V, E)$ be a graph. Let $f : E \to \{1, 2, ..., |E|\}$ be a perfect mapping. Then the map f is said to be a antimagic labeling of G if weight of all nodes are distinct, where weight of a node a is the sums of the labels of the arcs incident to the node a.

Definition 3.4 (Arumugam and Kamatchi [1]) Let $G = (V, E)$ be a graph. Let $f : V \to \{1, 2, ..., |V|\}$ be a perfect mapping.If $w(a) = \sum_{x \in N(a)} f(x)$ is the weight of the node a. Then the map f is said to be a distance antimagic labeling of G if all nodes have distinct weights. A graph that admits distance

antimagic labeling is called a distance antimagic graph. It is clear that if a graph contains two nodes with the same open neighborhood, then it does not admit a distance antimagic labeling.
Kamatchi and Arumugam [13] then conjectured that the converse of the previous statement is also true and proposed the following:
Conjecture 3.1 A graph G is distance antimagic if and only if G does not have two distinct nodes with the same open neighborhood.
Definition 3.5 (Arumugam et al. [2]) Let $G=(V,E)$ be a graph. Let $f:E\to\{1,2,...,|E|\}$ be a perfect mapping. Let $E(a)$ be a set of arcs incident to a node a. If $w(a)=\sum_{e\in E(a)} f(e)$ is the weight of the node a. Then the map f is said to be a local antimagic labeling of G if $w(a)\neq w(b)$ for all $ab\in E(G)$. Thus any local antimagic labeling induces a proper vertex coloring of G where the node a is assigned the color $w(a)$. The local antimagic chromatic number $\chi_{la}(G)$ is defined to be the minimum number of colors taken over all colorings of G induced by local antimagic labelings of G.
Definition 3.6 (Handa et al. [8]) Let $G=(V,E)$ be a graph. Let $f:V\to\{1,2,...,|V|\}$ be a perfect mapping. If $w(a)=\sum_{x\in N(a)} f(x)$ is the weight of the node a. Then the map f is said to be a local distance antimagic labeling of G if $w(a)\neq w(b)$ for all $ab\in E(G)$.
Definition 3.7 (Divya and Yamini [4]) The local distance antimagic chromatic number $\chi_{ld}(G)$ is defined as the minimum number of colors taken over all colorings induced by local distance antimagic labelings of G.
Motivated by these definitions, we introduce a concept named a local prime distance antimagic labeling of graphs. Also, a new parameter, its denotes $\chi_{lp}(G)$, named as local prime distance antimagic chromatic number.

LOCAL PRIME DISTANCE ANTIMAGIC LABELING:
This section defined that the local prime distance antimagic graphs and computed its properties for some classes of graphs.
Definition 4.1 Let $G=(V,E)$ be a graph. Let $f:V\to\{1,\ 2\ ...,\ |V|\}$ be a perfect mapping. If $w(t)=\sum_{a\in N(t)} f(a)$ is the weight of the node t. Then the map f is called the local prime distance antimagic labeling of G if for any $st\in E(G)$,

(i) $w(s)\neq w(t)$ and

(ii) $\gcd(f(s),f(t))=1$.

A graph G is local prime distance antimagic if G admits a local prime distance antimagic labeling. This induces a proper node coloring of the graph, with the node t assigned the color $w(t)$. This leads to the following definition.
Definition 4.2 The local prime distance antimagic chromatic number is defined as the minimum number of colors required to proper color the graph induced by local prime distance antimagic labeling of G and is denoted by $\chi_{lp}(G)$.

Here we compute the local prime distance antimagic chromatic number of star-related graphs. We know that the chromatic number of star graph $\chi(S_n)$ is 2. The following Theorem provides the local prime distance antimagic chromatic number of star graph $\chi_{lp}(S_n)$ is 2. Thus, we infer that $\chi_{lp}(S_n)=\chi(S_n)$.

Theorem 4.1 $\chi_{lp}(S_t)=2$.

Proof. Take G be a star graph S_t containing $t+1$ nodes. Let $V(G)=\{a,b_i:1\le i\le t\}$ and $E(G)=\{ab_i:1\le i\le t\}$ where a is the internal node and b_i are pendent nodes. Following that $|V(G)|=t+1$ and $|E(G)|=t$. We define a perfect mapping $f:V(G)\to\{1,2,...,t+1\}$ by

$$f(a)=1 \text{ and}$$
$$f(b_i)=i+1,\ 1\le i\le t.$$

Then $\gcd(f(a),f(b_i))=1$, $1\le i\le t$ and the node weights are as follows.

$$w(b_i)=f(a)=1 \text{ and}$$
$$\begin{aligned} w(a) &= f(b_1)+f(b_2)+...+f(b_t) \\ &= 2+3+...+(t+1) \\ &= \frac{(t+1)(t+2)}{2}-1. \end{aligned}$$

Clearly, $w(a)\ne w(b_i)$, $1\le i\le t$. Hence, $w(u)\ne w(v)$ for any two neighbourhood nodes u,v in G. Thus, $\chi_{lp}(S_t)=2$. Figure 1 shows the local prime distance antimagic chromatic number of S_t.

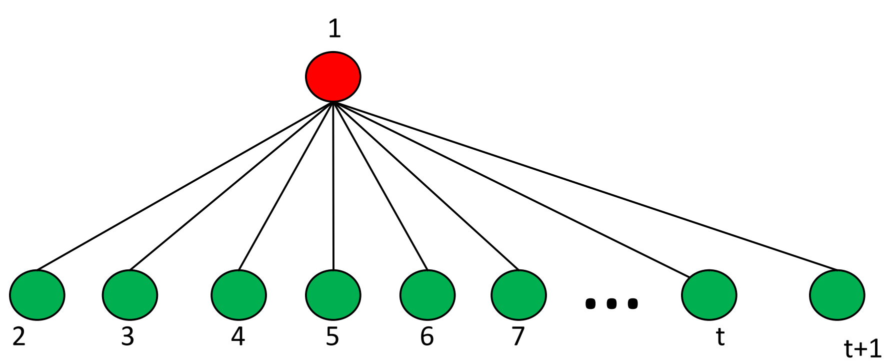

Figure 1 Local prime distance antimagic labeling of star S_t.

Theorem 4.2 $\chi_{lp}(S_{2,t})=t+2$ for $t\ge 3$.

Proof. Let G be a subdivision of star graph $S_{2,t}$ containing $2t+1$ nodes. . Let $V(G)=\{a,a_i^1,a_i^2:1\le i\le t\}$ and $E(G)=\{aa_i^1,a_i^1a_i^2:1\le i\le t\}$. Following that $|V(G)|=2t+1$ and $|E(G)|=2t$. We define a perfect mapping $f:V(G)\to\{1,2,...,2t+1\}$ by

$f(a)=1$ and

$f(a_i^j)=2i+j-1$, $1\le i\le t$, $j=1,2$.

Then $\gcd(f(a),f(a_i^1))=\gcd(f(a_i^1),f(a_i^2))=1$, $1\le i\le t$. Therefore, $\gcd(f(u),f(v))=1$ for any two neighbourhood nodes u,v in G. For the node weights we get

$W_1=w(a)=f(a_1^1)+f(a_2^1)+...+f(a_t^1)=2+4+6+...+2t=t(t+1)$

$W_2=\{w(a_i^1):1\le i\le t\}=\{f(a)+f(a_i^2):1\le i\le t\}=\{1+(2i+1):1\le i\le t\}=\{2i+2:1\le i\le t\}=\{4,6,8,...,2t+2\}$

$W_3=\{w(a_i^2):1\le i\le t\}=\{f(a_i^1):1\le i\le t\}=\{2i:1\le i\le t\}=\{2,4,6,...,2t\}$. Therefore,

$W=W_1\cup W_2\cup W_3=\{2,4,6,...,2t,2t+2,t(t+1)\}$.

For each i, since $2i+2\ne 2i$, $w(a_i^1)\ne w(a_i^2)$. Now, for any $t\ge 3$, the number $t(t+1)=t^2+t>2t+2$. This implies $w(a)>w(a_t^1)$. Thus, $w(a)>w(a_i^1)$, $i=1,2,...,t$. Therefore, $w(a)\ne w(a_i^1)$, $i=1,2,...,t$. This implies that, $|W|=t+2$ and $w(u)\ne w(v)$ for any $uv\in G$ and thus, $\chi_{lp}(S_{2,t})=t+2$ for $t\ge 3$. Figure 2 shows the local prime distance antimagic chromatic number of $S_{2,t}$.

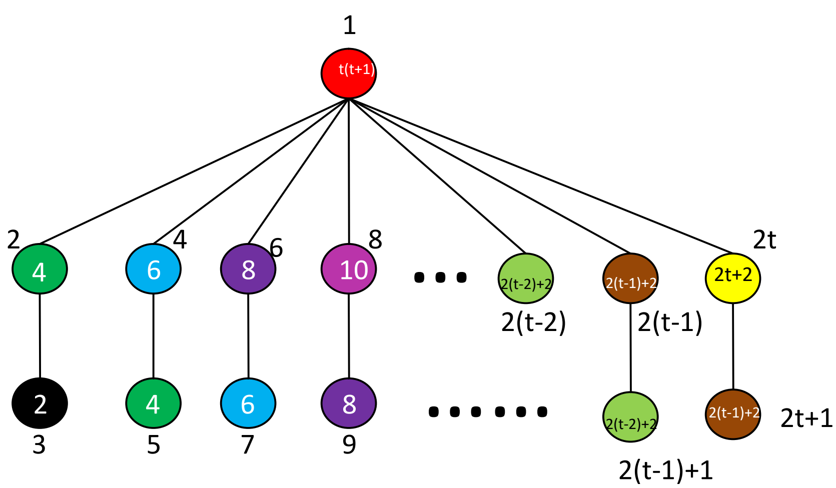

Figure 2 Local prime distance antimagic labeling of subdivision of star graph $S_{2,t}$.

Theorem 4.3 $\chi_{lp}(B_{t,t})=4$ for $t\ge 2$.

Proof Let G be a bistar graph $B_{t,t}$. Let $V(G)=\{a,b,a_i,b_i:1\le i\le t\}$ and $E(G)=\{ab,aa_i,bb_i:1\le i\le t\}$. Following that $|V(G)|=2(t+1)$ and $|E(G)|=2t+1$. We define a perfect mapping $f:V(G)\to\{1,2,...,2(t+1)\}$ by

$$f(a)=1,\ f(b)=2,\ f(a_i)=2i+2,\ 1\le i\le t \text{ and } f(b_i)=2i+1,\ 1\le i\le t.$$

Then we have,

i. $\gcd\left(f(a), f(b)\right) = \gcd(1,2) = 1$,

ii. $\gcd\left(f(a), f(a_i)\right) = \gcd(1, 2i+2) = 1,\ 1 \le i \le t$ and

iii. For $1 \le i \le t$, $\gcd\left(f(b), f(b_i)\right) = \gcd(2, 2i+1) = 1$ (since $2i+1$ is odd).

Therefore, $\gcd\left(f(u), f(v)\right) = 1$ for any two adjacent nodes u, v in G. The node weights are as follows:

$W_1 = w(a) = f(b) + f(a_1) + f(a_2) + ... + f(a_t) = 2+4+6+8+...+(2t+2) = (t+1)(t+2)$.

$W_2 = w(b) = f(a) + f(b_1) + f(b_2) + ... + f(b_t) = 1+3+5+7+...+(2t+1) = (t+1)^2$.

$W_3 = \left\{w(a_i) : 1 \le i \le t\right\} = \left\{f(a)\right\} = 1$.

$W_4 = \left\{w(b_i) : 1 \le i \le t\right\} = \left\{f(b)\right\} = 2$. Thus,

$W = W_1 \cup W_2 \cup W_3 \cup W_4 = \left\{1, 2, (t+1)(t+2), (t+1)^2\right\}$. That is, clearly, $|W| = 4$ and $w(u) \ne w(v)$ for any two adjacent nodes u, v in G. Hence $\chi_{lp}\left(B_{t,t}\right) = 4$ for $t \ge 2$. Figure 3 shows the local prime distance antimagic chromatic number of $B_{5,5}$.

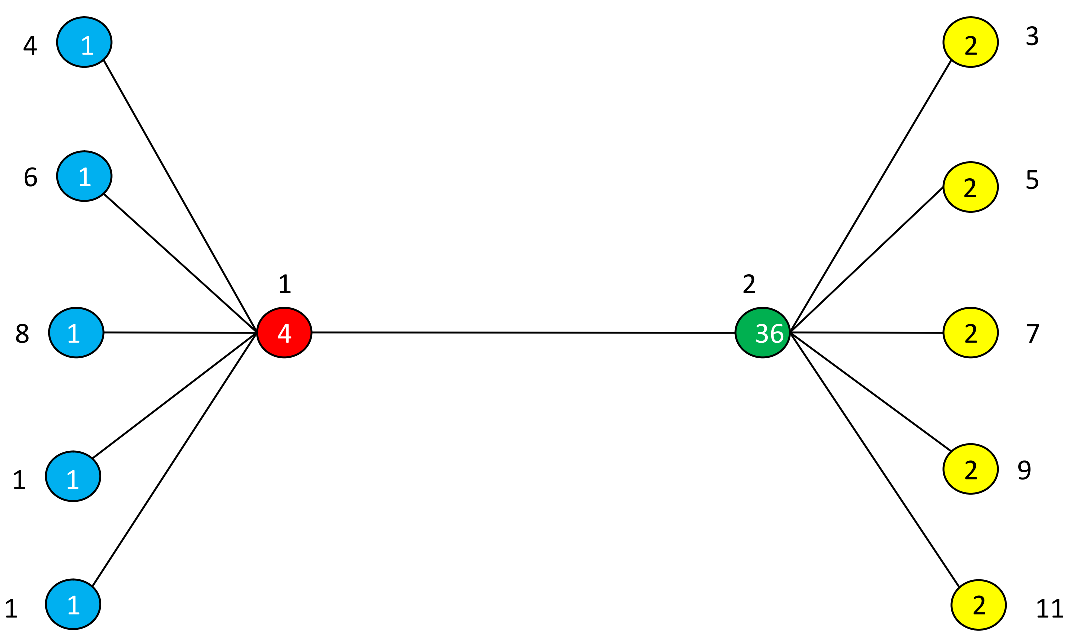

Figure 3 Local prime distance antimagic labeling of bistar graph $B_{5,5}$.

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Theorem 4.4 $\chi_{lp}\left(B_{s,t}\right) = 4$ *for* $s, t \ge 2$.

Proof. Let G be a double star graph $B_{s,t}$. Without loss of generality, we may assume that $s \le t$. Consider $V(G) = \left\{a, b, a_i, b_j : 1 \le i \le s, 1 \le j \le t\right\}$ and $E(G) = \left\{ab, aa_i, bb_j : 1 \le i \le s, 1 \le j \le t\right\}$. Thus $|V(G)| = s+t+2$ and $|E(G)| = s+t+1$. Let us define a perfect mapping $f : V(G) \to \left\{1, 2, ..., s+t+2\right\}$ by $f(a) = 1$, $f(b) = 2$, $f(a_i) = 2i+2\ (1 \le i \le s)$, $f(a_i) = i+s+2\ (s+1 \le i \le t)$ and $f(b_i) = 2i+1\ (1 \le i \le s)$.

Then prime labeling conditions are as follows:

i. $\gcd\left(f(a), f(b)\right) = \gcd(1,2) = 1$,

ii. For $1 \le i \le s$, $\gcd\left(f(a), f(a_i)\right) = \gcd(1, 2i+2) = 1$,

iii. For $s+1 \le i \le t$, $\gcd\left(f(a), f(a_i)\right) = \gcd(1, i+s+2) = 1$ and

iv. For $1 \le i \le s$, $\gcd\left(f(b), f(b_i)\right) = \gcd(2, 2i+1) = 1$ (since $2i+1$ is odd).

The node weights are as follows:

$$W_1 = w(a) = f(b) + f(a_1) + f(a_2) + \ldots + f(a_s) + f(a_{s+1}) + f(a_{s+2}) + \ldots + f(a_t)$$
$$= 2 + \left[4 + 6 + 8 + \ldots + (2s+2)\right] + \left[(2s+3) + (2s+4) + \ldots + (s+t+2)\right]$$
$$= \frac{n(1-n) + 4 + m(m+2n+5)}{2}.$$

$W_2 = w(b) = f(a) + f(b_1) + f(b_2) + \ldots + f(b_s) = 1 + 3 + 5 + 7 + \ldots + (2s+1) = (s+1)^2$.

$W_3 = \{w(a_i) : 1 \le i \le t\} = \{f(a)\} = 1$.

$W_4 = \{w(b_i) : 1 \le i \le s\} = \{f(b)\} = 2$. Thus we have,

$W = W_1 \cup W_2 \cup W_3 \cup W_4 = \left\{1, 2, (s+1)^2, \dfrac{n(1-n) + 4 + m(m+2n+5)}{2}\right\}$. That is, clearly, $|W| = 4$ and for any two adjacent nodes u, v in G, $w(u) \ne w(v)$. Hence $\chi_{lp}(B_{s,t}) = 4$ for $s, t \ge 2$.

Theorem 4.5 $\chi_{lp}\left(\langle B_{t,t} : w\rangle\right) = 5 \; for \; t \ge 2$.

Proof. Let G be a subdivision of a bistar graph $\langle B_{t,t} : w\rangle$. Let $V(G) = \{a, b, c, a_i, b_i : 1 \le i \le t\}$ and $E(G) = \{ac, bc, aa_i, bb_i : 1 \le i \le t\}$. Following that $|V(G)| = 2t+3$ and $|E(G)| = 2(t+1)$. Let $f : V(G) \to \{1, 2, \ldots, 2t+3\}$ be a perfect mapping defined by

$$f(a) = 1,\; f(b) = 2,\; f(c) = 3,\; f(a_i) = 2i+2 \text{ and } f(b_i) = 2i+3,\; 1 \le i \le t.$$

Then we have,

i. $\gcd\left(f(a), f(c)\right) = \gcd(1,3) = 1$,

ii. $\gcd\left(f(b), f(c)\right) = \gcd(2,3) = 1$,

iii. $\gcd\left(f(a), f(a_i)\right) = \gcd(1, 2i+2) = 1$, $1 \le i \le t$ and

iv. For $1 \le i \le t$, $\gcd\left(f(b), f(b_i)\right) = \gcd(2, 2i+3) = 1$ (since $2i+3$ is odd).

Thus for any two adjacent nodes u, v in G, the greatest common divisor is 1. Next, we have the node weights are

$W_1 = w(a) = f(c) + f(a_1) + f(a_2) + \ldots + f(a_t) = 3 + \left[4 + 6 + 8 + \ldots + (2t+2)\right] = 3 + (t+1)(t+2) - 2 = t^2 + 3t + 3$.

$W_2 = w(b) = f(c) + f(b_1) + f(b_2) + \ldots + f(b_t) = 3 + 5 + 7 + \ldots + (2t+3) = (t+2)^2 - 1 = t^2 + 4t + 3$.

$W_3 = w(c) = f(a) + f(b) = 1 + 2 = 3$.

$W_4 = \{w(a_i) : 1 \le i \le t\} = \{f(a)\} = 1$.

$W_5 = \{w(b_i):1 \le i \le t\} = \{f(b)\} = 2$.

Hence, $|W| = 5$ where $W = W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5 = \{1,2,3,t^2+3t+3,t^2+4t+3\}$ and also clearly, $w(u) \ne w(v)$ for any $uv \in E(G)$. Thus, $\chi_{lp}(\langle B_{t,t} : w \rangle) = 5$ for $t \ge 2$.

Theorem 4.6 $\chi_{lp}\left(Fr_t^{(3)}\right) = 2t+1 \, for \; t \ge 2$.

Proof. If G is a friendship graph $Fr_t^{(3)}$. Taking $V(G) = \{a,b_i,c_i:1 \le i \le t\}$ and $E(G) = \{ab_i,ac_i,b_ic_i:1 \le j \le t\}$. We have $|V(G)| = 2t+1$ and $|E(G)| = 3t$. Define a perfect mapping $f:V(G) \to \{1,2,...,2t+1\}$ by

$$f(a)=1,\ f(b_i)=2i,\ 1 \le i \le t, \text{ and } f(c_i)=2i+1,\ 1 \le i \le t.$$

Then prime labeling conditions are as follows:

i. For $1 \le i \le t$, $\gcd(f(a),f(b_i)) = \gcd(1,2i) = 1$,
ii. For $1 \le i \le t$, $\gcd(f(a),f(c_i)) = \gcd(1,2i+1) = 1$ and
iii. For $1 \le i \le t$, $\gcd(f(b_i),f(c_i)) = \gcd(2i,2i+1) = 1$ (since $2i$ and $2i+1$ are consecutive integers).

The node weights are as follows:

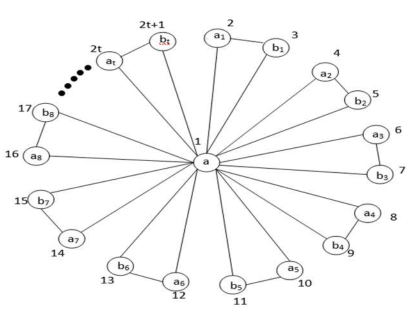
$$W_1 = w(a) = f(b_1)+f(b_2)+...+f(b_t)+f(c_1)+f(c_2)+...+f(c_t)$$
$$=[2+4+6+8+...+2t]+[3+5+7+...+(2t+1)]$$
$$=\frac{(2t+1)(2t+2)}{2}-1=\frac{2t(2t+3)}{2}=t(2t+3).$$

$W_2 = \{w(b_i):1 \le i \le t\} = \{f(a)+f(c_i):1 \le i \le t\} = \{1+(2i+1):1 \le i \le t\} = \{2i+2:1 \le i \le t\} = \{4,6,8,...,2t+2\}$.

$W_3 = \{w(c_i):1 \le i \le t\} = \{f(a)+f(b_i):1 \le i \le t\} = \{1+2i:1 \le i \le t\} = \{3,5,7,...,2t+1\}$.

Therefore, $W = W_1 \cup W_2 \cup W_3 = \{t(2t+3),2i+1,2i+2:1 \le i \le t\}$. For every i, suppose that $2i+1 = 2i+2$. Then $1 = 2$, which is not possible. Thus we have $w(b_i) \ne w(c_i)$, That is, clearly, $w(b_i) \ne w(a) \ne w(c_i)$ because $t(2t+3) > 2t+1$ and $t(2t+3) > 2t+2$.

Hence $|W| = 2t+1$ and $w(u) \ne w(v)$ for any two neighborhood nodes u,v in G and thus, $\chi_{lp}\left(Fr_t^{(3)}\right) = 2t+1$ for $t \ge 2$. Figure 4 shows the local prime distance antimagic graph for $Fr_t^{(3)}$.



Theorem 4.7 $\chi_{lp}\left(Fr_t^{(4)}\right)=2t+1$ *for* $t\geq 2$.

Proof. Let G be a friendship graph $Fr_t^{(4)}$. Taking $V(G)=\{a,b_i,c_i,d_i:1\leq i\leq t\}$ and $E(G)=\{ab_i,ac_i,b_id_i,c_id_i:1\leq j\leq t\}$. We have $|V(G)|=3t+1$ and $|E(G)|=4t$. Let $f:V(G)\to\{1,2,...,2t+1\}$ be a perfect mapping defined by

$$f(a)=1,\ f(b_i)=3(i-1)+2,\ 1\leq i\leq t,\ f(c_i)=3(i-1)+4,\ 1\leq i\leq t \text{ and } f(d_i)=3i,\ 1\leq i\leq t.$$

Then for any two adjacent nodes u,v in G, $\gcd(f(u),f(v))=1$ because we have,

i. For $1\leq i\leq t$, $\gcd(f(a),f(b_i))=\gcd(1,3(i-1)+2)=1$,
ii. For $1\leq i\leq t$, $\gcd(f(a),f(c_i))=\gcd(1,3(i-1)+4)=1$ and
iii. For $1\leq i\leq t$, $\gcd(f(b_i),f(d_i))=\gcd(3(i-1)+2,3i)=\gcd(3i-1,3i)=1$ (since $3i$ and $3i-1$ are consecutive integers).
iv. For $1\leq i\leq t$, $\gcd(f(b_i),f(d_i))=\gcd(3(i-1)+4,3i)=\gcd(3i+1,3i)=1$ (since $3i$ and $3i+1$ are consecutive integers).

The node weights are following in the way:

$$W_1=w(a)=f(b_1)+f(b_2)+...+f(b_t)+f(c_1)+f(c_2)+...+f(c_t)$$
$$=\left[2+5+8+...+(3(t-1)+2)\right]+\left[4+7+10+...+(3(t-1)+4)\right]$$
$$=(1+2+3+...+(3t+1))-1-(3+6+9+...+3t)=\frac{(3t+1)(3t+2)}{2}-1-\frac{3t(t+1)}{2}=3t(t+1).$$
$$W_2=\{w(b_i):1\leq i\leq t\}=\{f(a)+f(d_i):1\leq i\leq t\}=\{3i+1:1\leq i\leq t\}=\{4,7,10,...,3t+1\}.$$
$$W_3=\{w(c_i):1\leq i\leq t\}=\{f(a)+f(d_i):1\leq i\leq t\}=\{1+3i:1\leq i\leq t\}=\{4,7,10,...,3t+1\}.$$
$$W_4=\{w(d_i):1\leq i\leq t\}=\{f(b_i)+f(c_i):1\leq i\leq t\}=\{(3(i-1)+2)+(3(i-1)+4):1\leq i\leq t\}$$
$$=\{6i:1\leq i\leq t\}=\{6,12,18,...,6t\}.$$

For every i, suppose that $6i=3i+1$. Then $i=\frac{1}{3}$, which is not possible. Thus we have $w(b_i)\neq w(d_i)\neq w(c_i)$ .Also, $w(a)$ is greater than $w(b_i)$, $w(c_i)$ and $w(d_i)$ for all i, because $3t(t+1)>3t+1$ and $3t(t+1)>6t$. Hence $w(u)\neq w(v)$ for all $uv\in E(G)$.

Thus $|W|=2t+1$ because $W=W_1\cup W_2\cup W_3\cup W_4=\{3t(t+1),3i+1,6i:1\leq i\leq t\}$. Hence $\chi_{lp}\left(Fr_t^{(4)}\right)=2t+1$ for $t\geq 2$.

Theorem 4.8 $\chi_{lp}(W_t)=t-1$ for t is even and $t\geq 4$.

Proof. Consider G is a wheel graph W_t. Let $V(G)=\{a,a_i:1\leq i\leq t\}$ and $E(G)=\{aa_i,a_ja_{j+1},a_ta_1:1\leq i\leq t,1\leq j\leq t-1\}$. Following that $|V(G)|=t+1$ and $|E(G)|=2t$. We define a perfect mapping $f:V(G)\to\{1,2,...,t+1\}$ by

$$f(a)=1, \text{ and } f(a_i)=i+1,\ 1\leq i\leq t.$$

Then it is noted that

i. For $1 \le i \le t$, $\gcd(f(a), f(a_i)) = \gcd(1, i+1) = 1$,

ii. For $1 \le i \le t-1$, $\gcd(f(a_i), f(a_{i+1})) = \gcd(i+1, i+2) = 1$,(since $i+1$ and $i+2$ are consecutive integers) and

iii. $\gcd(f(a_t), f(a_1)) = \gcd(t+1, 2) = 1$ (since $t+1$ is odd).

The node weights are as follows:

$W_1 = w(a) = f(a_1) + f(a_2) + ... + f(a_t) = 2 + 3 + 4 + ... + t = \frac{t(t+1)}{2} - 1 = \frac{t^2 + t - 2}{2}$,

$W_2 = w(a_1) = f(a) + f(a_t) + f(a_2) = 1 + (t+1) + 3 = t + 5$,

$W_3 = \{w(a_i) : 2 \le i \le t-1\} = \{f(a) + f(a_{i-1}) + f(a_{i+1}) : 2 \le i \le t-1\} = \{1 + i + (i+2) : 2 \le i \le t-1\}$
$= \{2i + 3 : 2 \le i \le t-1\} = \{7, 9, 11, ..., (2t+1)\}$.

$W_4 = w(a_t) = f(a) + f(a_1) + f(a_{t-1}) = 1 + 2 + t = t + 3$.

Obviously, $w(a) > w(a_i) \forall i \Rightarrow w(a) \ne w(a_i) \forall i$. If $i = \frac{t}{2}$, then $w\left(a_{\frac{t}{2}}\right) = 2\frac{t}{2} + 3 = t + 3 = w(a_t)$. Also, if $i = \frac{t}{2} + 1$, then $w\left(a_{\frac{t}{2}+1}\right) = 2\left(\frac{t}{2} + 1\right) + 3 = t + 5 = w(a_1)$. Therefore, $w(u) \ne w(v)$ for any two adjacent nodes u, v in G.

Thus, we have $W = W_1 \cup W_2 \cup W_3 \cup W_4 = \left\{\frac{t^2 + t - 2}{2}, 2i + 3 : 2 \le i \le t-1\right\}$. This implies, $|W| = t - 1$ and hence, $\chi_{lp}(W_t) = t - 1$ for t is even and $t \ge 4$.

Corrolary 4.1 W_t is not a local prime distance antimagic graph for t is odd and $t \ge 4$.

Theorem 4.9 $\chi_{lp}(F_t) = t + 1$ where t is odd, $t \ge 5$ and $t = 4$.

Proof. Let G be a fan graph F_t. Let $V(G) = \{a, a_i : 1 \le i \le t\}$ and $E(G) = \{aa_i, a_j a_{j+1} : 1 \le i \le t, 1 \le j \le t-1\}$. Following that $|V(G)| = t + 1$ and $|E(G)| = 2t - 1$. We define a perfect mapping $f : V(G) \to \{1, 2, ..., t+1\}$ by

$f(a) = 1$, and

$f(a_i) = i + 1$, $1 \le i \le t$.

Then prime labeling conditions are as follows:

i. For $1 \le i \le t$, $\gcd(f(a), f(a_i)) = \gcd(1, i+1) = 1$ and

ii. For $1 \le i \le t-1$, $\gcd(f(a_i), f(a_{i+1})) = \gcd(i+1, i+2) = 1$, (since $i+1$ and $i+2$ are consecutive integers).

The node weights are as follows:

$W_1 = w(a) = f(a_1) + f(a_2) + ... + f(a_t) = 2 + 3 + 4 + ... + (t+1) = \frac{(t+1)(t+2)}{2} - 1 = \frac{t(t+3)}{2}$,

$W_2 = w(a_1) = f(a) + f(a_2) = 1 + 3 = 4$,

$W_3 = \{w(a_i): 2 \le i \le t-1\} = \{f(a)+f(a_{i-1})+f(a_{i+1}): 2 \le i \le t-1\} = \{2i+3: 2 \le i \le t-1\} = \{7,9,11,...,(2t+1)\}$.

$W_4 = w(a_t) = f(a)+f(a_{t-1}) = t+1$.

$W = W_1 \cup W_2 \cup W_3 \cup W_4 = \left\{\frac{t(t+3)}{2}, 4, t+1, 2i+3: 2 \le i \le t-1\right\}$. Suppose that $w(a) = w(a_t)$. Then $\frac{t(t+3)}{2} = t+1 \Rightarrow t = 1, t = -2$ which is not possible because $t \ge 4$. Now,

$$\begin{aligned} w(a) &= \frac{t(t+3)}{2} \\ &= \frac{t^2+3t}{2} + (2t+1) - (2t+1) \\ &= \frac{t^2-t-2}{2} + (2t+1) \\ &> 2t+1 \\ &= w(a_{t-1}). \end{aligned}$$

Thus, for each i, $2 \le i \le t-1$, $w(a) \ne w(a_i)$. Also, for $t \ge 4$, $w(a) = \frac{t(t+3)}{2} > 4 = w(a_1)$. This implies $w(a) \ne w(a_1)$. If $t(\ge 5)$ is odd, then $w(a_t)$ is even and $w(a_i)$ is odd, $2 \le i \le t-1$. Also, if $t = 4$, then $w(a_t) < w(a_i)$, $2 \le i \le t-1$. Hence $w(a_t) \ne w(a_i)$, $2 \le i \le t-1$. Obviously, $w(a_1) \ne w(a_i)$, $2 \le i \le t$. Thus, $|W| = t+1$ and also, for any two adjacent nodes u, v in G, $w(u) \ne w(v)$. Therefore, $\chi_{lp}(F_t) = t+1$ where t is odd, $t \ge 5$ and $t = 4$.

Theorem 4.10 $\chi_{lp}(F_t) = t$ where t is even, $t \ge 6$ and $t = 3$.

Proof. Consider G is a fan graph F_t. Let $V(G) = \{a, a_i: 1 \le i \le t\}$ and $E(G) = \{aa_i, a_j a_{j+1}: 1 \le i \le t, 1 \le j \le t-1\}$. Then $|V(G)| = t+1$ and $|E(G)| = 2t-1$. We define a perfect mapping $f: V(G) \to \{1, 2, ..., t+1\}$ by

$f(a) = 1$, and

$f(a_i) = i+1$, $1 \le i \le t$.

Then prime labeling conditions are as follows:

i. For $1 \le i \le t$, $\gcd(f(a), f(a_i)) = \gcd(1, i+1) = 1$ and
ii. For $1 \le i \le t-1$, $\gcd(f(a_i), f(a_{i+1})) = \gcd(i+1, i+2) = 1$, (since $i+1$ and $i+2$ are consecutive integers).

The node weights are as follows:

$W_1 = w(a) = f(a_1)+f(a_2)+...+f(a_t) = 2+3+4+...+(t+1) = \frac{(t+1)(t+2)}{2} - 1 = \frac{t(t+3)}{2}$,

$W_2 = w(a_1) = f(a)+f(a_2) = 1+3 = 4$,

$W_3 = \{w(a_i): 2 \le i \le t-1\} = \{f(a) + f(a_{i-1}) + f(a_{i+1}): 2 \le i \le t-1\} = \{2i+3: 2 \le i \le t-1\} = \{7, 9, 11, ..., (2t+1)\}$.

$W_4 = w(a_t) = f(a) + f(a_{t-1}) = t+1$.

Suppose that $w(a) = w(a_t)$. Then $\frac{t(t+3)}{2} = t+1 \Rightarrow t = 1, t = -2$ which is contradiction to the assumption of t. Now,

$$\begin{aligned} w(a) &= \frac{t(t+3)}{2} \\ &= \frac{t^2+3t}{2} + (2t+1) - (2t+1) \\ &= \frac{t^2-t-2}{2} + (2t+1) \\ &> 2t+1 \\ &= w(a_{t-1}). \end{aligned}$$

Thus, for every i, $2 \le i \le t-1$, $w(a) \ne w(a_i)$. Also, for $t \ge 3$, $w(a) = \frac{t(t+3)}{2} > 4 = w(a_1)$. This implies $w(a) \ne w(a_1)$. Suppose, for $2 \le i \le t-1$, $w(a_i) = w(a_1)$. Then $2i+3 = 4 \Rightarrow i = \frac{1}{2}$ which is not possible. Thus, $w(a_i) \ne w(a_1)$. Also, we have $w(a_i) = w(a_t)$ where $t = 2i+2$ and $w(a_1) = w(a_t)$ where $t = 3$. Thus $w(u) \ne w(v)$ for any two adjacent nodes u, v in G.

Also, $W = W_1 \cup W_2 \cup W_3 \cup W_4 = \left\{\frac{t(t+3)}{2}, 4, 2i+3: 2 \le i \le t-1\right\}$. This implies $|W| = t$. Hence, $\chi_{lp}(F_t) = t$ where t is even, $t \ge 6$ and $t = 3$.

CONCLUSION:

In this paper, the concept of local prime distance antimagic chromatic number of a graph is introduced. We found local distance antimagic labelings for several families of graphs including the stars, subdivision of stars, bistar graphs, double star graphs, subdivision of a bistar graphs, friendship graphs, wheel graphs and fan graphs. We also suggest the following open problem.

Open problem 1 Determine $\chi_{lp}(P_n)$, $n > 1$.

Open problem 2 Determine $\chi_{lp}(K_{2,n})$, $n > 2$.

Open problem 3 Determine $\chi_{lp}(P_n \times P_2)$, $n \ge 2$.

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